

THE PROBABILITY DENSITY CURVE

The molecules of the atmosphere have many parameters. Some of these parameters have a wide distribution of values. With some parameters, this distribution of values tends to cluster about the mean value. When these parameter values are randomly distributed, the farther that a value departs from the mean value, the less likely it is to occur.

Definition of the Probability Density Curve: The distribution of values in a random distribution can be expressed by a probability density equation of the form:

$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{(x - \bar{x})^2}{2\sigma^2} \quad \text{PDC01}$$

Here, $\Phi(x)$ is the probability that the parameter will have the value x , \bar{x} is the mean value of the parameter for the population, and σ is the standard deviation of the distribution.

Definition of the Mean Value: We can further define the mean value \bar{x} as the sum of all the values divided by the number of values; i. e.,

$$\bar{x} = \frac{1}{n} \sum_1^n x \quad \text{PDC02}$$

Here, n is the total number of individual values of x .

Definition of the Standard Deviation: The standard deviation σ is defined as the square-root of the mean of the squares of the individual deviations from the mean.

$$\sigma = \left(\frac{1}{n} \sum_1^n (x - \bar{x})^2 \right)^{\frac{1}{2}} \quad \text{PDC03}$$

The standard deviation always has a positive value, since negative values of $x - \bar{x}$ are removed in the squaring process.

History of the Probability Density Curve: The concept was discovered by Abraham de Moivre in 1733 and used in his *Doctrine of Chances*. The concept was enlarged and extended by Pierre-Simon Laplace in 1812 in his *Analytical Theory of Probabilities*. At roughly the same time, Karl-Friedrich Gauss claimed to have been using the concept since 1794 and published a dissertation on the normal error distribution in astronomical observations in 1809. Properly speaking, the curve should only be referred to as Gaussian only when it refers to errors in measurements.

The Normal Distribution Curve: When the distribution mean is set to zero and the standard deviation is set to one, the probability density curve is said to be “normalized”. It becomes the normal distribution curve:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \exp - \frac{x^2}{2} \quad \text{PDC04}$$

The normal distribution curve may be written in many forms. In the form shown above, the constant forces the area under the curve to be equal to unity, and x is expressed in terms of the standard deviation of the distribution. Note that this equation is only valid when the mean is equal to zero. Note also that the equation is independent of the value of the standard deviation, as long as x is always expressed in terms of that standard deviation.

Molecular Speed Distribution Along a Single Cartesian Axis: In [statistical mechanics](#), the curve can be used to describe the distribution of molecular speeds along any single axis of the standard tri-axial orthogonal Cartesian reference system for an ideal gas under conditions of equilibrium. The curve then takes the form:

$$\Phi(v_{\pm x}) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{v_x^2}{2\sigma^2} \quad \text{PDC05}$$

Here, $\Phi(x)$ is the probability that a molecule chosen at random from the total population will have the speed v_x . The speed may be in either the positive direction ($+x$) or in the negative direction ($-x$).

This expression is valid for any single axis, no matter what the orientation or location of the axis might be within the population. Obviously, then, it is also valid for any combination of such axes, such as make up our [proximity axis](#) in one-dimensional space.

Proximity Speed Distribution Along the Positive Axial Arm: When applied to the positive arm of the proximity speed axis in one-dimensional space, the expression takes the form

$$\Phi(v_p) = \frac{2}{\sigma\sqrt{2\pi}} \exp - \frac{v_p^2}{2\sigma^2} \quad \text{PDC06}$$

Here, $\Phi(v_p)$ tells us the probability of a molecule moving normal to and toward the surface of interest having speed v_p . The factor of 2 in the constant is necessary because the total probability must remain at unity and we are here only dealing with the half the total population that lies along the positive arm of the proximity axis.

These equations will tell us the probability that an individual molecule will have some specific speed. However, what if we want to know what proportion of the population has speeds between two specific values? For that, we turn to other members of the family of probability density curves: the error function curve and the complementary error function curve.

The Error Function Curve: The error function of value x , $erf(x)$, is usually given by the normalized expression:

$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{x^2}{2}\right) dx \quad \text{PDC07}$$

The function returns the proportion of the area under the normal curve (the total area being unity) that lies between the value minus infinity and the value x expressed in standard deviations.

Proximity Speed Error Function: For the range of proximity speeds, Equation PDC07 must be modified to:

$$erf(v_p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v_p} \exp\left(-\frac{v_p^2}{2}\right) dv_p \quad \text{PDC08}$$

This expression give the proportion of molecules having proximity speeds lying between minus infinity and v_p . All speeds are expressed in multiples of the standard deviation speed.

The Complementary Error Function Curve: The complimentary error function of value x , $erfc(x)$, is usually given by the normalized expression:

$$erfc(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad \text{PDC09}$$

The function returns the proportion of the area under the normal curve (the total area being unity) that lies between the value infinity and the value x expressed in standard deviations.

Proximity Speed Complementary Error Function: For the range of proximity speeds, Equation PDC09 must be modified to:

$$erfc(v_p) = \frac{1}{\sqrt{2\pi}} \int_{v_p}^{\infty} \exp\left(-\frac{v_p^2}{2}\right) dv_p \quad \text{PDC10}$$

This expression gives the proportion of molecules having proximity speeds lying between v_p and infinity. All velocities are expressed in multiples of the standard deviation speed.

Evaluating the Error Function: These expressions cannot be evaluated explicitly, but can be solved numerically. For those of my readers who wish to construct algorithms for the expressions, Sergei Winitzki proposed (and David Cantrell modified) the following approximation. The approximation is said to be correct to about five significant figures over the entire range of the expression. I offer it here without endorsement, as I have not evaluated it personally.

$$erf(x) \approx \left[1 - \exp\left(-x^2 \frac{4 + ax^2}{1 + ax^2}\right) \right]^{\frac{1}{2}}$$

where the constant a is 0.147 (Cantrell's improvement).

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REFERENCES

Internal References: References to other essays in this collection are linked in the essay text by hyperlinks. You may follow these hyperlinks or ignore them, as you choose.

External References: These are papers by other authors that contain statements or data that are specifically incorporated into this essay.

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